

ON THE BLOW UP CRITERION OF 3D-NSE IN SOBOLEV-GEVREY SPACES

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ABSTRACT. In [5], Benameur proved a blow-up result of the non regular solution of (NSE) in the Sobolev-Gevrey spaces. In this paper we improve this result, precisely we give an exponential type explosion in Sobolev-Gevrey spaces with less regularity on the initial condition. Fourier analysis is used.

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1. INTRODUCTION

The 3D incompressible Navier-Stokes equations are given by:

$$(NSE) \quad \begin{cases} \partial_t u - \nu \Delta u + u \cdot \nabla u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0, x) = u^0(x) & \text{in } \mathbb{R}^3, \end{cases}$$

where ν is the viscosity of fluid, $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and $(u \cdot \nabla u) := u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$, while $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is a given initial velocity. If u^0 is quite regular, the divergence free condition determines the pressure p .

Our problem is the study of explosions for non-smooth solutions of (NSE). In the literature, several authors studied this problem (see [2, 3, 4, 7]), however all the obtained results do not exceed $(T^* - t)^{-\sigma}$, where T^* denotes the maximal time of existence. Recently, in [5], the author gives a positif answer on the question: Is the type of explosion is due to

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the chosen space or the nonlinear part of (NSE)? For this, he used the Sobolev-Gevrey spaces which are defined as follows; for $a, s \geq 0$ and $\sigma \geq 1$,

$$H_{a,\sigma}^s(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3); e^{a|D|^{1/\sigma}} f \in H^s(\mathbb{R}^3)\}$$

is equipped with the norm

$$\|f\|_{H_{a,\sigma}^s(\mathbb{R}^3)}^2 = \|e^{a|D|^{1/\sigma}} f\|_{H^s}^2$$

and its associated inner product

$$\langle f/g \rangle_{H_{a,\sigma}^s(\mathbb{R}^3)} = \langle e^{a|D|^{1/\sigma}} f / e^{a|D|^{1/\sigma}} g \rangle_{H^s(\mathbb{R}^3)}.$$

Precisely, for $a > 0$, $s > 3/2$, $\sigma > 1$ and $u^0 \in (H_{a,\sigma}^s(\mathbb{R}^3))^3$ such that $\operatorname{div} u^0 = 0$. He proved that, there is a unique time $T^* \in (0, \infty]$ and a unique solution $u \in \mathcal{C}([0, T^*], H_{a,\sigma}^s(\mathbb{R}^3))$ to (NSE) system such that $u \notin \mathcal{C}([0, T^*], H_{a,\sigma}^s(\mathbb{R}^3))$. Moreover, if $T^* < \infty$, then

$$(1.1) \quad c(T^* - t)^{-s/3} \exp\left(aC(T^* - t)^{-\frac{1}{3\sigma}}\right) \leq \|u(t)\|_{H_{a,\sigma}^s}, \quad \forall t \in [0, T^*),$$

where $c = c(s, u^0, \sigma) > 0$ and $C = C(s, u^0, \sigma) > 0$. The choice of this spaces is due to the scaling property: If $u(t, x)$ is a solution to (NSE) system with the initial data $u^0(x)$, then for any $\lambda > 0$, $\lambda u(\lambda^2 t, \lambda x)$ is a solution to (NSE) system with the initial data $\lambda u^0(\lambda x)$. Also, we recall that the classical used spaces $\mathbf{B}(\mathbb{R}^3)$ for (NSE) system satisfy the fundamental condition:

$$u^0(x) \in \mathbf{B}(\mathbb{R}^3) \iff \lambda u^0(\lambda x) \in \mathbf{B}(\mathbb{R}^3), \quad \forall \lambda > 0,$$

which is not valid in the Sobolev-Gevrey spaces $H_{a,\sigma}^s(\mathbb{R}^3)$.

Our work is intended to improve the result in [5]. Before generalizing this result, we recall the energy estimate: If $u \in \mathcal{C}([0, T], H^s(\mathbb{R}^3))$, with $s > 5/2$, is a solution to (NSE), then

$$(1.2) \quad \|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(z)\|_{L^2}^2 dz = \|u^0\|_{L^2}^2.$$

Now we are ready to state our main result.

Theorem 1.1. *Let $a > 0$ and $\sigma > 1$. Let $u^0 \in (H_{a,\sigma}^1(\mathbb{R}^3))^3$ be such that $\operatorname{div} u^0 = 0$, then there is a unique $T^* \in (0, \infty]$ and unique $u \in \mathcal{C}([0, T^*], H_{a,\sigma}^1(\mathbb{R}^3))$ solution to (NSE) system such that $u \notin \mathcal{C}([0, T^*], H_{a,\sigma}^1(\mathbb{R}^3))$. If $T^* < \infty$, then*

$$(1.3) \quad \frac{c_1}{(T^* - t)^{\frac{2\sigma_0+1}{3\sigma} + \frac{1}{3}}} \exp\left[\frac{ac_2}{(T^* - t)^{1/3\sigma}}\right] \leq \|u(t)\|_{H_{a,\sigma}^1},$$

where $c_1 = c_1(u^0, a, \sigma) > 0$, $c_2 = c_2(u^0, \sigma) > 0$ and $2\sigma_0$ is the integer part of 2σ .

The remainder of this paper is organized in the following way: In section 2, we give some notations and important preliminary results. Section 3 is devoted to prove that (NSE) is well posed in $H_{a,\sigma}^1(\mathbb{R}^3)$. In section 4, we prove the exponential type explosion of non regular solution to (NSE) system.

2. NOTATIONS AND PRELIMINARY RESULTS

2.1. Notations. In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix \cdot \xi) f(x) dx, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

- The inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(-i\xi \cdot x) g(\xi) d\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

- For $s \in \mathbb{R}$, $H^s(\mathbb{R}^3)$ denotes the usual non-homogeneous Sobolev space on \mathbb{R}^3 and $\langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^3)}$ denotes the usual scalar product on $H^s(\mathbb{R}^3)$.

- For $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^3)$ denotes the usual homogeneous Sobolev space on \mathbb{R}^3 and $\langle \cdot, \cdot \rangle_{\dot{H}^s(\mathbb{R}^3)}$ denotes the usual scalar product on $\dot{H}^s(\mathbb{R}^3)$.

- The convolution product of a suitable pair of function f and g on \mathbb{R}^3 is given by

$$(f * g)(x) := \int_{\mathbb{R}^3} f(y) g(x - y) dy.$$

- If $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ are two vector fields, we set

$$f \otimes g := (g_1 f, g_2 f, g_3 f),$$

and

$$\operatorname{div}(f \otimes g) := (\operatorname{div}(g_1 f), \operatorname{div}(g_2 f), \operatorname{div}(g_3 f)).$$

- For $a > 0$, $\sigma \geq 1$, we denote the Sobolev-Gevrey space defined as follows

$$\dot{H}_{a,\sigma}^1(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3); e^{a|D|^{1/\sigma}} f \in \dot{H}^1(\mathbb{R}^3)\}$$

which is equipped with the norm

$$\|f\|_{\dot{H}_{a,\sigma}^1} = \|e^{a|D|^{1/\sigma}} f\|_{\dot{H}^1}$$

and the associated inner product

$$\langle f/g \rangle_{\dot{H}_{a,\sigma}^1} = \langle e^{a|D|^{1/\sigma}} f / e^{a|D|^{1/\sigma}} g \rangle_{\dot{H}^1}.$$

2.2. Preliminary results. In this section we recall some classical results and we give new technical lemmas.

Lemma 2.1. (See [1]) Let $(s, t) \in \mathbb{R}^2$, such that $s < \frac{3}{2}$ and $s + t > 0$. Then there exists a constant C , such that

$$\|uv\|_{\dot{H}^{s+t-\frac{3}{2}}} \leq C(\|u\|_{\dot{H}^s} \|v\|_{\dot{H}^t} + \|u\|_{\dot{H}^t} \|v\|_{\dot{H}^s}).$$

If $s < \frac{3}{2}$, $t < \frac{3}{2}$ and $s + t > 0$, then there exists a constant C , such that

$$\|uv\|_{\dot{H}^{s+t-\frac{3}{2}}} \leq C\|u\|_{\dot{H}^s} \|v\|_{\dot{H}^t}.$$

Lemma 2.2. (See [5]) For $\delta > 3/2$, we have

$$\|\widehat{f}\|_{L^1(\mathbb{R}^3)} \leq C_\delta \|f\|_{L^2}^{1-\frac{3}{2\delta}} \|f\|_{\dot{H}^\delta(\mathbb{R}^3)}^{\frac{3}{2\delta}},$$

with $C_\delta = 2\sqrt{\frac{\pi}{3}}\left((\frac{2\delta}{3}-1))^{3/(4\delta)} + (\frac{2\delta}{3}-1)^{-1+\frac{3}{4\delta}}\right)$.

Moreover, for all $\delta_0 > 3/2$, there is $M(\delta_0) > 0$ such that

$$C_\delta \leq M(\delta_0), \quad \forall \delta \geq \delta_0.$$

Lemma 2.3. *For every $a > 0$, $\sigma \geq 1$, and for every $f, g \in \dot{H}_{a,\sigma}^1(\mathbb{R}^3)$, we have $fg \in \dot{H}_{a,\sigma}^1(\mathbb{R}^3)$ and*

$$\|fg\|_{\dot{H}_{a,\sigma}^1(\mathbb{R}^3)} \leq 16 \left(\|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} f)\|_{L^1} \|g\|_{\dot{H}_{a,\sigma}^1} + \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} g)\|_{L^1} \|f\|_{\dot{H}_{a,\sigma}^1} \right)$$

Proof of the lemma 2.3. We have

$$\begin{aligned} \|fg\|_{\dot{H}_{a,\sigma}^1(\mathbb{R}^3)}^2 &= \int_{\xi} |\xi|^2 e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{fg}(\xi)|^2 \\ &= \int_{\xi} |\xi|^2 e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f} * \widehat{g}(\xi)|^2 \\ &\leq \int_{\xi} |\xi|^2 e^{2a|\xi|^{\frac{1}{\sigma}}} (|\widehat{f}| * |\widehat{g}|)^2(\xi) \\ &\leq \int_{\xi} \left(\int_{\eta} |\xi| e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi - \eta)| |\widehat{g}(\eta)| \right)^2 \\ &\leq \int_{\xi} \left(\int_{|\eta| < |\xi - \eta|} |\xi| e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi - \eta)| |\widehat{g}(\eta)| + \int_{|\eta| > |\xi - \eta|} |\xi| e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi - \eta)| |\widehat{g}(\eta)| \right)^2. \end{aligned}$$

By the elementary inequality

$$(1 + b)^\theta \leq 1 + \theta b^\theta, \quad \forall b, \theta \in [0, 1],$$

we can deduce that

$$|\xi|^{\frac{1}{\sigma}} \leq \max(|\xi - \eta|, |\eta|)^{\frac{1}{\sigma}} + \frac{1}{\sigma} \min(|\xi - \eta|, |\eta|)^{\frac{1}{\sigma}}, \quad \forall \xi, \eta \in \mathbb{R}^3.$$

Therefore

$$\begin{aligned} \|fg\|_{\dot{H}_{a,\sigma}^1(\mathbb{R}^3)}^2 &\leq 4 \int_{\xi} (I_1 + I_2)^2 d\xi \\ &\leq 16 \int_{\xi} I_1^2(\xi) + I_2^2(\xi) d\xi, \end{aligned}$$

where

$$\begin{aligned} I_1(\xi) &= \int_{\eta} |\xi - \eta| e^{a|\xi - \eta|^{\frac{1}{\sigma}}} |\widehat{f}(\xi - \eta)| e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}} |\widehat{g}(\eta)| \\ I_2(\xi) &= \int_{\eta} e^{\frac{a}{\sigma}|\xi - \eta|^{\frac{1}{\sigma}}} |\widehat{f}(\xi - \eta)| e^{a|\eta|^{\frac{1}{\sigma}}} |\eta| |\widehat{g}(\eta)|. \end{aligned}$$

Using the following notations:

$$\begin{aligned} \alpha_1(\xi) &= |\xi| e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)| \\ \alpha_2(\xi) &= e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)| \\ \beta_1(\xi) &= |\xi| e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{g}(\xi)| \\ \beta_2(\xi) &= e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{g}(\xi)|, \end{aligned}$$

we obtain

$$\|fg\|_{\dot{H}_{a,\sigma}^1(\mathbb{R}^3)}^2 \leq 16(\|\alpha_1 * \beta_2\|_{L^2}^2 + \|\alpha_2 * \beta_1\|_{L^2}^2).$$

Young's inequality yields

$$\|fg\|_{\dot{H}_{a,\sigma}^1(\mathbb{R}^3)}^2 \leq 16(\|\alpha_1\|_{L^2}^2 \|\beta_2\|_{L^1}^2 + \|\alpha_2\|_{L^1}^2 \|\beta_1\|_{L^2}^2),$$

and the proof of lemma 2.3 is finished.

Lemma 2.4. *Let $a > 0$, $\sigma \geq 1$ and $s \geq 1$, then there is a constant $c = c(s, a, \sigma)$ such that for all $f \in \dot{H}_{a,\sigma}^1(\mathbb{R}^3)$*

$$(2.1) \quad \|f\|_{\dot{H}^s} \leq c\|f\|_{\dot{H}_{a,\sigma}^1}.$$

Proof of the lemma 2.4. Let $k_0 \in \mathbb{Z}^+$ such that

$$\frac{k_0}{2\sigma} \leq s - 1 < \frac{k_0 + 1}{2\sigma}.$$

Using the fact that

$$|\xi|^{2s-2} \leq |\xi|^{\frac{k_0}{\sigma}} + |\xi|^{\frac{k_0+1}{\sigma}}, \quad \forall \xi \in \mathbb{R}^3,$$

we get

$$\|f\|_{\dot{H}^s}^2 \leq \frac{(k_0 + 1)!}{(2a)^{k_0+1}} (2a+1) \int_{\mathbb{R}^3} \left(\frac{(2a|\xi|^{1/\sigma})^{k_0}}{k_0!} + \frac{(2a|\xi|^{1/\sigma})^{k_0+1}}{(k_0 + 1)!} \right) |\xi|^2 |\widehat{f}(\xi)|^2 \leq 2 \frac{(k_0 + 1)!}{(2a)^{k_0}} \|f\|_{\dot{H}_{a,\sigma}^1}^2.$$

Lemma 2.5. *For every $a > 0$, $\sigma \geq 1$, and for every $f, g \in \dot{H}_{a,\sigma}^1(\mathbb{R}^3)$, we have $fg \in \dot{H}_{a,\sigma}^1(\mathbb{R}^3)$ and*

$$(2.2) \quad \|fg\|_{L^2(\mathbb{R}^3)} \leq C \left(\|f\|_{\dot{H}_{a,\sigma}^1} \|g\|_{L^2} + \|g\|_{\dot{H}_{a,\sigma}^1} \|f\|_{L^2} \right).$$

Proof of the lemma 2.5. We have

$$\|fg\|_{L^2(\mathbb{R}^3)} \leq C \left(\|f\|_{L^\infty(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^\infty(\mathbb{R}^3)} \right).$$

Using the classical interpolation inequality

$$\|h\|_{L^\infty} \leq c\|h\|_{\dot{H}^1}^{1/2} \|h\|_{\dot{H}^2}^{1/2}$$

and lemma 2.4, we can deduce inequality (2.2).

Remark 2.6. By lemmas 2.3-2.5, we can deduce that $\dot{H}_{a,\sigma}^1(\mathbb{R}^3)$ is an algebra for $a > 0$ and $\sigma > 1$.

Lemma 2.7. *If $f \in \dot{H}_{a,\sigma}^1(\mathbb{R}^3)$, then*

$$(2.3) \quad \|f\|_{\dot{H}_{a,\sigma}^1}^2 \leq 2(e^{2a} + 1) \left(\|f\|_{L^2}^2 + \|f\|_{\dot{H}_{a,\sigma}^1}^2 \right) \leq 4(e^{2a} + 1) \|f\|_{\dot{H}_{a,\sigma}^1}^2.$$

Proof of the lemma 2.7. Simply write the integral as a sum of low frequencies $\{\xi \in \mathbb{R}^3; |\xi| < 1\}$ and high frequencies $\{\xi \in \mathbb{R}^3; |\xi| > 1\}$.

3. WELL-POSEDNESS OF (NSE) IN $H_{a,\sigma}^1(\mathbb{R}^3)$

In the following theorem, we characterize the existence and uniqueness of the solution.

Theorem 3.1. *Let $a > 0$ and $\sigma \geq 1$. Let $u^0 \in (H_{a,\sigma}^1(\mathbb{R}^3))^3$ such that $\operatorname{div} u^0 = 0$. Then, there exist a unique $T^* \in (0, +\infty]$ and a unique solution $u \in \mathcal{C}([0, T^*], H_{a,\sigma}^1(\mathbb{R}^3))$ to (NSE) system such that $u \notin \mathcal{C}([0, T^*], H_{a,\sigma}^1(\mathbb{R}^3))$. Moreover, if T^* is finite, then*

$$(3.1) \quad \limsup_{t \nearrow T^*} \|u(t)\|_{\dot{H}_{a,\sigma}^1} = \infty.$$

Proof theorem 3.1. The integral form of Navier-Stokes equations is

$$u = e^{\nu t \Delta} u^0 + B(u, u),$$

where

$$B(u, v) = - \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(\operatorname{div}(u \otimes v))$$

and

$$\mathcal{F}(\mathbb{P}f)(\xi) = \widehat{f}(\xi) - \frac{(f(\xi) \cdot \xi)}{|\xi|^2} \xi.$$

The main idea of this proof is to apply the theorem of the fixed point to the operator

$$u \mapsto e^{\nu t \Delta} u^0 + B(u, u).$$

Taking account of lemma 2.7, it suffices to estimate the nonlinear part respectively in the spaces $L^2(\mathbb{R}^3)$ and $\dot{H}_{a,\sigma}^1(\mathbb{R}^3)$.

- To estimate $B(u, v)$ in $\dot{H}_{a,\sigma}^1(\mathbb{R}^3)$, we have

$$\begin{aligned} \|B(u, v)\|_{\dot{H}_{a,\sigma}^1}(t) &\leq \int_0^t \|e^{\nu(t-\tau) \Delta} \mathbb{P}(\operatorname{div}(u \otimes v))\|_{\dot{H}_{a,\sigma}^1} \\ &\leq \int_0^t \left(\int_{\xi} |\xi|^3 e^{-2\nu(t-\tau)|\xi|^2} |\xi| e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{u * v}(\tau, \xi)|^2 \right)^{\frac{1}{2}} \\ &\leq \int_0^t \frac{1}{\nu^{\frac{3}{4}}(t-\tau)^{\frac{3}{4}}} \|u \otimes v\|_{\dot{H}_{a,\sigma}^{\frac{1}{2}}} \end{aligned}$$

with

$$\begin{aligned} \|u \otimes v\|_{\dot{H}_{a,\sigma}^{\frac{1}{2}}}^2(\tau) &= \int_{\xi} |\xi| e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{u * v}(\tau, \xi)|^2 \\ &\leq \int_{\xi} |\xi| \left(\int_{\eta} e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\tau, \xi - \eta)| |\widehat{v}(\tau, \eta)| \right)^2. \end{aligned}$$

Using the inequality $e^{a|\xi|^{\frac{1}{\sigma}}} \leq e^{a|\xi-\eta|^{\frac{1}{\sigma}}} e^{a|\eta|^{\frac{1}{\sigma}}}$, we get

$$\begin{aligned} \|u \otimes v\|_{\dot{H}_{a,\sigma}^{\frac{1}{2}}}^2(\tau) &\leq \int_{\xi} |\xi| \left(\int_{\eta} e^{a|\xi-\eta|^{\frac{1}{\sigma}}} |\widehat{u}(\tau, \xi - \eta)| e^{a|\eta|^{\frac{1}{\sigma}}} |\widehat{v}(\tau, \eta)| \right)^2 \\ &\leq \|UV\|_{\dot{H}^{\frac{1}{2}}}^2 \\ &\leq c\|U\|_{\dot{H}^1}\|V\|_{\dot{H}^1} \quad (\text{by lemma 2.1}) \\ &\leq c\|u\|_{\dot{H}_{a,\sigma}^1}\|v\|_{\dot{H}_{a,\sigma}^1}, \end{aligned}$$

where $\widehat{U} = e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\tau, \xi)|$ and $\widehat{V} = e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{v}(\tau, \xi)|$.
Therefore,

$$(3.2) \quad \|B(u, v)\|_{\dot{H}_{a,\sigma}^1(t)} \leq c\nu^{-\frac{3}{4}} T^{\frac{1}{4}} \|u\|_{L_T^\infty(\dot{H}_{a,\sigma}^1)} \|v\|_{L_T^\infty(\dot{H}_{a,\sigma}^1)}.$$

• To estimate $B(u, v)$ in $L^2(\mathbb{R}^3)$, we have

$$\begin{aligned} \|B(u, v)\|_{L^2(t)} &\leq \int_0^t \|e^{\nu(t-\tau)\Delta} \mathbb{P}(\operatorname{div}(u \otimes v))\|_{L^2} \\ &\leq \int_0^t \left(\int_\xi e^{-2\nu(t-\tau)|\xi|^2} |\xi|^2 |\widehat{u * v}(\tau, \xi)|^2 \right)^{\frac{1}{2}} \\ &\leq \int_0^t \frac{1}{\nu^{\frac{1}{4}}(t-\tau)^{\frac{1}{4}}} \|u \otimes v\|_{\dot{H}^{\frac{1}{2}}} \\ &\leq \int_0^t \frac{1}{\nu^{\frac{1}{4}}(t-\tau)^{\frac{1}{4}}} \|u\|_{\dot{H}^1} \|v\|_{\dot{H}^1} \quad (\text{by lemma 2.1}), \\ &\leq \int_0^t \frac{1}{\nu^{\frac{1}{4}}(t-\tau)^{\frac{1}{4}}} \|u\|_{\dot{H}_{a,\sigma}^1} \|v\|_{\dot{H}_{a,\sigma}^1}, \\ &\leq \nu^{-\frac{1}{4}} t^{\frac{3}{4}} \|u\|_{\dot{H}_{a,\sigma}^1} \|v\|_{\dot{H}_{a,\sigma}^1}. \end{aligned}$$

Therefore,

$$(3.3) \quad \|B(u, v)\|_{L^2(t)} \leq c\nu^{-\frac{1}{4}} T^{\frac{3}{4}} \|u\|_{L_T^\infty(\dot{H}_{a,\sigma}^1)} \|v\|_{L_T^\infty(\dot{H}_{a,\sigma}^1)}.$$

Now, we recall the following fixed point theorem.

Lemma 3.2. (See [6]) Let X be an abstract Banach space equipped with the norm $\|\cdot\|$ and let $B : X \times X \rightarrow X$ be a bilinear operator, such that for any $x_1, x_2 \in X$,

$$\|B(x_1, x_2)\| \leq c_0 \|x_1\| \|x_2\|.$$

Then, for any $y \in X$ such that

$$4c_0 \|y\| < 1$$

the equation

$$x = y + B(x, x)$$

has a solution $x \in X$. In particular, this solution satisfies

$$\|x\| \leq 2\|y\|$$

and it is the only one such that

$$\|x\| < \frac{1}{2c_0}.$$

Combining the inequalities (3.2)-(3.3) and lemma 3.2, and choosing a time T small enough, we guarantee the existence and uniqueness of the solution in the space $\mathcal{C}([0, T], H_{a,\sigma}^1(\mathbb{R}^3))$.

• Now, we want to prove the inequality (3.2) by contradiction. Let u be maximal solution to (NSE) with $T^* < \infty$. It suffices to observe that u is bounded, then we can extend the solution to an interval $[0, T_1]$, where $T_1 > T^*$.

As the function u is assumed to be bounded, then there exists $M \geq 0$ such that

$$\|u(t)\|_{H_{a,\sigma}^1(\mathbb{R}^3)} \leq M, \quad \forall t \in [0, T^*].$$

We deduce that, $\forall \varepsilon > 0 \exists \alpha / \forall t, t' < T^* \text{ such that } T^* - t < \alpha \text{ and } T^* - t' < \alpha$, we have

$$\|u(t) - u(t')\|_{\dot{H}_{a,\sigma}^1(\mathbb{R}^3)} < \varepsilon.$$

Then $(u(t))$ is a Cauchy sequence in $H_{a,\sigma}^1(\mathbb{R}^3)$ at the time T^* . The space $H_{a,\sigma}^1(\mathbb{R}^3)$ is complete, then there exists $u^* \in H_{a,\sigma}^1(\mathbb{R}^3)$ such that

$$\lim_{t \nearrow T^*} u(t) = u^* \text{ in } H_{a,\sigma}^1(\mathbb{R}^3).$$

Consider the following system

$$(NSE^*) \quad \begin{cases} \partial_t v - \nu \Delta v + v \cdot \nabla v &= -\nabla p' \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ v(0) &= u^* \text{ in } \mathbb{R}^3. \end{cases}$$

Applying the first step to (NSE^*) system, then there exist $T_1 > 0$ and a unique solution $v \in \mathcal{C}([0, T_1], H_{a,\sigma}^1(\mathbb{R}^3))$ to (NSE^*) . So

$$w(t) = \begin{cases} u(t) & \text{if } t \in [0, T^*] \\ v(t - T^*) & \text{if } t \in [T^*, T^* + T_1] \end{cases}$$

as a unique solution extends u , which is absurd. Taking into account the energy inequality (1.2) we can deduce equation (3.1). Therefore theorem 3.1 is proved.

4. PROOF OF THE MAIN RESULT

4.1. Fundamental inequalities. In this section, we give some blow up results which are necessary to prove our main result.

Theorem 4.1. *Let $u^0 \in (H_{a,\sigma}^1(\mathbb{R}^3))^3$ such that $\operatorname{div} u^0 = 0$ and $u \in \mathcal{C}([0, T^*], H_{a,\sigma}^1(\mathbb{R}^3))$ the maximal solution to (NSE) system given by theorem 1.1. If $T^* < \infty$, then*

$$(4.1) \quad \int_t^{T^*} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u(\tau))\|_{L^1}^2 d\tau = \infty, \quad \forall t \in [0, T^*)$$

and

$$(4.2) \quad \frac{(\frac{\nu}{2})^{\frac{1}{2}}}{(T^* - t)^{\frac{1}{2}}} \leq \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1}, \quad \forall t \in [0, T^*).$$

Proof of theorem 4.1. • First, we wish to prove the inequality (4.1).

Taking the scalar product in $\dot{H}_{a,\sigma}^1(\mathbb{R}^3)$, we obtain

$$\begin{aligned} \frac{1}{2} \partial_t \|u(t)\|_{\dot{H}_{a,\sigma}^1}^2 + \nu \|\nabla u(t)\|_{\dot{H}_{a,\sigma}^1}^2 &\leq |\langle e^{a|D|^{\frac{1}{\sigma}}} (u \cdot \nabla u) / e^{a|D|^{\frac{1}{\sigma}}} u(t) \rangle_{\dot{H}^1}| \\ &\leq |\langle e^{a|D|^{\frac{1}{\sigma}}} \operatorname{div} (u \otimes u) / e^{a|D|^{\frac{1}{\sigma}}} u(t) \rangle_{\dot{H}^1}| \\ &\leq |\langle e^{a|D|^{\frac{1}{\sigma}}} u \otimes u / \nabla e^{a|D|^{\frac{1}{\sigma}}} u(t) \rangle_{\dot{H}^1}| \\ &\leq \|u \otimes u\|_{\dot{H}_{a,\sigma}^1} \|\nabla u\|_{\dot{H}_{a,\sigma}^1}. \end{aligned}$$

Using lemma 2.3, we have

$$\frac{1}{2} \partial_t \|u(t)\|_{\dot{H}_{a,\sigma}^1}^2 + \nu \|\nabla u(t)\|_{\dot{H}_{a,\sigma}^1}^2 \leq 32 \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u(t))\|_{L^1} \|u(t)\|_{\dot{H}_{a,\sigma}^1} \|\nabla u(t)\|_{\dot{H}_{a,\sigma}^1}.$$

Inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$ gives us

$$\frac{1}{2} \partial_t \|u(t)\|_{\dot{H}_{a,\sigma}^1}^2 + \nu \|\nabla u(t)\|_{\dot{H}_{a,\sigma}^1}^2 \leq c\nu^{-1} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u(t))\|_{L^1}^2 \|u(t)\|_{\dot{H}_{a,\sigma}^1}^2 + \frac{\nu}{2} \|\nabla u(t)\|_{\dot{H}_{a,\sigma}^1}^2.$$

Then

$$\partial_t \|u(t)\|_{\dot{H}_{a,\sigma}^1}^2 + \nu \|\nabla u(t)\|_{\dot{H}_{a,\sigma}^1}^2 \leq c\nu^{-1} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u(t))\|_{L^1}^2 \|u(t)\|_{\dot{H}_{a,\sigma}^1}^2.$$

Gronwall lemma implies that, for every $0 \leq t \leq T < T^*$,

$$\|u(T)\|_{\dot{H}_{a,\sigma}^1}^2 \leq \|u(t)\|_{\dot{H}_{a,\sigma}^1}^2 e^{c\nu^{-1} \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u(\tau))\|_{L^1}^2 d\tau}.$$

The fact that $\limsup_{t \nearrow T^*} \|u(t)\|_{\dot{H}_{a,\sigma}^1} = \infty$ implies that

$$\int_t^{T^*} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u(\tau))\|_{L^1}^2 d\tau = \infty, \quad \forall t \in [0, T^*).$$

• Now, we want to prove inequality (4.2).

Returning to (NSE) system and taking the Fourier transform in the first equation, we get

$$\frac{1}{2} \partial_t |\widehat{u}(t, \xi)|^2 + \nu |\xi|^2 |\widehat{u}(t, \xi)|^2 + \operatorname{Re}(\mathcal{F}(u \cdot \nabla u)(t, \xi) \cdot \widehat{u}(t, -\xi)) = 0.$$

But for any $\varepsilon > 0$,

$$\frac{1}{2} \partial_t |\widehat{u}(t, \xi)|^2 = \frac{1}{2} \partial_t (|\widehat{u}(t, \xi)|^2 + \varepsilon) = \sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon} \cdot \partial_t \sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon}$$

and

$$\partial_t \sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon} + \nu |\xi|^2 \frac{|\widehat{u}(t, \xi)|^2}{\sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon}} + \frac{\mathcal{F}(u \cdot \nabla u)(t, \xi) \cdot \widehat{u}(t, -\xi)}{\sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon}} = 0.$$

Then, we have

$$\partial_t \sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon} + \nu |\xi|^2 \frac{|\widehat{u}(t, \xi)|^2}{\sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon}} \leq |\mathcal{F}(u \cdot \nabla u)(t, \xi)|.$$

Integrating on $[t, T] \subset [0, T^*)$, we obtain

$$\sqrt{|\widehat{u}(T, \xi)|^2 + \varepsilon} + \nu |\xi|^2 \int_t^T \frac{|\widehat{u}(\tau, \xi)|^2}{\sqrt{|\widehat{u}(\tau, \xi)|^2 + \varepsilon}} d\tau \leq \sqrt{|\widehat{u}(t, \xi)|^2 + \varepsilon} + \int_t^T |\mathcal{F}(u \cdot \nabla u)(\tau, \xi)| d\tau.$$

Taking $\varepsilon \rightarrow 0$, we get

$$\begin{aligned} |\widehat{u}(T, \xi)| + \nu |\xi|^2 \int_t^T |\widehat{u}(\tau, \xi)| d\tau &\leq |\widehat{u}(t, \xi)| + \int_t^T |\mathcal{F}(u \cdot \nabla u)(\tau, \xi)| d\tau \\ &\leq |\widehat{u}(t, \xi)| + \int_t^T |\widehat{u}| *_{\xi} |\widehat{\nabla u}|(\tau, \xi) d\tau \\ &\leq |\widehat{u}(t, \xi)| + \int_t^T \int_{\eta} |\widehat{u}(\tau, \xi - \eta)| |\widehat{\nabla u}(\tau, \eta)| d\eta d\tau. \end{aligned}$$

Multiplying the late equation by $e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}$ and using the inequality

$$e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} \leq e^{\frac{a}{\sigma}|\xi - \eta|^{\frac{1}{\sigma}}} e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}},$$

we obtain

$$\begin{aligned} &e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(T, \xi)| + \nu |\xi|^2 \int_t^T e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\tau, \xi)| d\tau \\ &\leq e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(t, \xi)| + \int_t^T \int_{\eta} e^{\frac{a}{\sigma}|\xi - \eta|^{\frac{1}{\sigma}}} |\widehat{u}(\tau, \xi - \eta)| e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}} |\widehat{\nabla u}(\tau, \eta)| d\eta d\tau. \end{aligned}$$

Integrating over $\xi \in \mathbb{R}^3$ and using Young's inequality, we get

$$\begin{aligned} & \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(T)\|_{L^1} + \nu \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} \Delta u)\|_{L^1} \\ & \leq \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1} + \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)\|_{L^1} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} \nabla u)\|_{L^1}. \end{aligned}$$

Cauchy-Schwarz inequality yields

$$\begin{aligned} & \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(T)\|_{L^1} + \nu \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} \Delta u)\|_{L^1} \\ & \leq \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1} + \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)\|_{L^1}^{\frac{3}{2}} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} \Delta u)\|_{L^1}^{\frac{1}{2}}. \end{aligned}$$

Inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$ gives

$$\|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(T)\|_{L^1} + \frac{\nu}{2} \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} \Delta u)\|_{L^1} \leq \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1} + \frac{1}{\nu} \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)\|_{L^1}^3.$$

By the Gronwall lemma, for $0 \leq t \leq T < T^*$, we obtain

$$\|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(T)\|_{L^1} \leq \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1} e^{\nu^{-1} \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(\tau)\|_{L^1}^2 d\tau}$$

or

$$\|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(T)\|_{L^1}^2 e^{-2\nu^{-1} \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(\tau)\|_{L^1}^2 d\tau} \leq \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1}^2.$$

Integrating over $[t_0, T] \subset [0, T^*)$, we get

$$1 - e^{-2\nu^{-1} \int_t^T \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(\tau)\|_{L^1}^2 d\tau} \leq 2\nu^{-1} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t_0)\|_{L^1}^2 (T - t_0).$$

By inequality (4.1), if $T \rightarrow T^*$, we have

$$1 \leq 2\nu^{-1} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t_0)\|_{L^1}^2 (T^* - t_0).$$

Then we can deduce inequality (4.2), and theorem 4.1 is proved.

4.2. Exponential type explosion. Let $a > 0$ and $\sigma > 1$. Let $u \in \mathcal{C}([0, T^*), H_{a,\sigma}^1(\mathbb{R}^3))$ be a maximal solution to (NSE) system given by theorem 3.1 and suppose that $T^* < \infty$. We want to prove inequality (1.3).

- Firstly, we prove the following result

$$(4.3) \quad \frac{\nu^{\frac{2}{\sigma}}}{T^* - t} \leq \|\widehat{u}(t)\|_{L^1}^2, \quad \forall t \in [0, T^*).$$

For this, using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|\mathcal{F}(e^{\frac{a}{\sigma}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1} &= \int_{\xi} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi)| \\ &\leq c_{a,\sigma} \|u\|_{\dot{H}_{\frac{a}{\sqrt{\sigma}}, \sigma}^1} \end{aligned}$$

where

$$c_{a,\sigma}^2 = \int_{\mathbb{R}^3} \frac{1}{|\xi|^2} e^{-2a(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma})|\xi|^{\frac{1}{\sigma}}} = 4\pi\sigma \left(2a(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma})\right)^{\sigma-2} \Gamma(\sigma) < \infty.$$

Since $\frac{a}{\sqrt{\sigma}} < a$, then $\dot{H}_{a,\sigma}^1(\mathbb{R}^3) \hookrightarrow \dot{H}_{\frac{a}{\sqrt{\sigma}},\sigma}^1(\mathbb{R}^3)$.

Let's take $a' = \frac{a}{\sqrt{\sigma}}$, we ensure that $u \in \mathcal{C}([0, T^*), \dot{H}_{a',\sigma}^1(\mathbb{R}^3))$ and

$$\limsup_{t \nearrow T^*} \|u(t)\|_{\dot{H}_{a',\sigma}^1} = \infty.$$

Using similar technique, we get

$$\int_t^{T^*} \|\mathcal{F}(e^{\frac{a'}{\sqrt{\sigma}}|D|^{\frac{1}{\sigma}}} u(\tau))\|_{L^1}^2 d\tau = \infty, \quad \forall t \in [0, T^*)$$

and

$$\frac{\frac{\nu}{2}}{T^* - t} \leq \|\mathcal{F}(e^{\frac{a'}{\sqrt{\sigma}}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1}^2, \quad \forall t \in [0, T^*).$$

This implies

$$\frac{\frac{\nu}{2}}{T^* - t} \leq \|\mathcal{F}(e^{\frac{a}{(\sqrt{\sigma})^2}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1}^2, \quad \forall t \in [0, T^*).$$

By induction, we can deduce that for every $n \in \mathbb{N}$,

$$\frac{\frac{\nu}{2}}{T^* - t} \leq \|\mathcal{F}(e^{\frac{a}{(\sqrt{\sigma})^n}|D|^{\frac{1}{\sigma}}} u)(t)\|_{L^1}^2, \quad \forall t \in [0, T^*).$$

Using dominated convergence theorem, we get inequality (1.3) by letting $n \rightarrow +\infty$.

- Secondly, we prove the exponential type explosion. Using lemma 2.2 and the energy estimate

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(z)\|_{L^2}^2 dz \leq \|u^0\|_{L^2}^2,$$

we can write, for $k \in \mathbb{N}$ such that $1 + \frac{k}{2\sigma} \geq 2$ (i.e $k \geq 2\sigma$),

$$\frac{\sqrt{\frac{\nu}{2}}}{\sqrt{T^* - t}} \leq \|\widehat{u}(t)\|_{L^1} \leq M(2) \|u^0\|_{L^2}^{1 - \frac{3}{2(1 + \frac{k}{2\sigma})}} \|u(t)\|_{\dot{H}^{1 + \frac{k}{2\sigma}}}^{\frac{3}{2(1 + \frac{k}{2\sigma})}}, \quad \forall t \in [0, T^*).$$

Then

$$\left(\frac{\sqrt{\frac{\nu}{2}}}{\sqrt{T^* - t}}\right)^{2(\frac{1 + \frac{k}{2\sigma}}{3})} (M(2))^{-2(\frac{1 + \frac{k}{2\sigma}}{3})} \leq \|u^0\|_{L^2}^{2(\frac{1 + \frac{k}{2\sigma}}{3}) - 1} \|u(t)\|_{\dot{H}^{1 + \frac{k}{2\sigma}}}$$

or

$$\left(\frac{\sqrt{\frac{\nu}{2}}}{\sqrt{T^* - t}}\right)^{2(\frac{1 + \frac{k}{2\sigma}}{3})} (M(2))^{-2(\frac{1 + \frac{k}{2\sigma}}{3})} \|u^0\|_{L^2}^{1 - 2(\frac{1 + \frac{k}{2\sigma}}{3})} \leq \|u(t)\|_{\dot{H}^{1 + \frac{k}{2\sigma}}}.$$

We obtain

$$\frac{C_1}{(T^* - t)^{2/3}} \left(\frac{C_2}{(T^* - t)^{1/3\sigma}}\right)^k \leq \|u(t)\|_{\dot{H}^{1 + \frac{k}{2\sigma}}}^2$$

with

$$C_1 = \left(\frac{\nu}{2}(M(2))^{-2} \|u^0\|_{L^2}\right)^{\frac{2}{3}}, \quad C_2 = \left(\frac{\nu}{2}(M(2))^{-2} \|u^0\|_{L^2}^2\right)^{\frac{1}{3\sigma}}.$$

Then

$$\frac{1}{k!} \frac{C_1}{(T^* - t)^{2/3}} \left(\frac{2aC_2}{(T^* - t)^{1/3\sigma}}\right)^k \leq \int_{\xi} \frac{(2a)^k}{k!} |\xi|^{\frac{k}{\sigma}} |\xi|^2 |\widehat{u}(t, \xi)|^2.$$

Summing over the set $\{k \in \mathbb{N}; k \geq 2\sigma\}$, we get

$$\frac{C_1}{(T^* - t)^{1/3}} \left(e^{\frac{2aC_2}{(T^* - t)^{1/3\sigma}}} - \sum_{0 \leq k \leq 2\sigma} \frac{(\frac{2aC_2}{(T^* - t)^{1/3\sigma}})^k}{k!}\right) \leq \int_{\xi} \left(e^{2a|\xi|^{\frac{1}{\sigma}}} - \sum_{0 \leq k \leq 2\sigma} \frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!}\right) |\xi|^2 |\widehat{u}(t, \xi)|^2$$

$$\leq \int_{\xi} e^{2a|\xi|^{\frac{1}{\sigma}}} |\xi|^2 |\widehat{u}(t, \xi)|^2.$$

Now, put

$$h(z) = \frac{e^z - \sum_{k=0}^{2\sigma_0} \frac{z^k}{k!}}{z^{2\sigma_0+1} e^{\frac{z}{2}}}, \quad z > 0,$$

with $2\sigma_0$ is the integer part of 2σ .

The function h satisfies the following properties

- h is continuous on $(0, \infty)$,
- $h(z) > 0$ for all $z > 0$,
- $\lim_{z \rightarrow \infty} h(z) = \infty$,
- $\lim_{z \rightarrow 0^+} h(z) = \frac{1}{(2\sigma_0 + 1)!}$.

Then there is $B = B(\sigma_0) > 0$ such that $h(z) \geq B$ for all $z > 0$. Therefore

$$\frac{c_1}{(T^* - t)^{\frac{1}{3} + \frac{2\sigma_0+1}{3\sigma}}} \exp \left[\frac{aC_2}{(T^* - t)^{1/3\sigma}} \right] \leq \|u(t)\|_{H_{a,\sigma}^1},$$

with $c_1 = BC_1(2aC_2)^{2\sigma_0+1}$. Then, we can deduce inequality (1.3) and theorem 1.1 is proved.

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